Continuity of the trace operator with respect to the domain and application to shape optimization

A. Boulkhemair[1], A. Chakib[2] and A. Nachaoui[1]

[1] Laboratoire de Mathématiques Jean Leray,
CNRS UMR6629/ Université de Nantes,
2, rue de la Houssinière, BP 92208, 44322 Nantes, France.
E-mail: boulkhem@math.univ-nantes.fr & nachaoui@math.univ-nantes.fr

[2] Département de Mathématiques Appliquées et Informatique
FST de Beni-Mellal, Université Cadi-Ayyad
B.P. 523, Beni-Mellal, Morocco.
E-mail: chakib@fstbm.ac.ma

Abstract

Shape optimization amounts to find the optimal shape of a domain which minimizes a given criterion, often called a cost functional. Here, we are interested in the case where the criterion is computed through the solution of a partial differential equation, the so-called state equation, which makes the optimization problem non-trivial. We use a general parameterization of the unknown boundary in order to preserve the physical general information and we prove the continuity of the cost functional.

1 Introduction

Varying domains constitute an important type of inverse problems which appear in connection with a variety of phenomenon in different fields. Shape Identification, free or moving boundary problems, phase change or elastic contact problems fall into the class of such inverse problems with a priory unknown domains. Some problems that lead to boundary inverse problems are: the dam seepage problem [14]; Riabouchinsky flow past circular disks [21]; Falling droplets [35, 32, 38]; the Alt-Caffarelli or Bernoulli problem [4]. The common structure of these problems is that there are elliptic equations for m unknowns and $m + 1$ mixed Dirichlet-Neumann conditions on the unknown surface. Many more problems also fall into this general category, with applications to aluminium production by electrolytic reduction [13]; groundwater flow [25, 29]; free surface waves [9]]; semiconductors [2, 16]; electromagnetic casting [33, 31]; the Dam and Bernoulli problems [10, 37, 28, 22, 17].

There is no previous analytic theory covering this general class of boundary inverse problems, although there has been progress on individual problems.

A variety of techniques exist for numerically solving boundary inverse problems, see [14, 26, 16, 1, 11] and the references therein. Among this, shape optimization technique is a useful method for solving the type of problems mentioned above. This is a variational approach, minimizing an integral functional over the variable domain having an unknown boundary. For an introduction to shape optimization, cf [23, 36] and some examples of its use are given in [5, 8, 15, 18, 34].

A general formulation of such optimal shape approaches can be expressed as follows

$$\begin{cases} \text{Minimize } J(\Omega, u) \\ \text{subject to } \Omega \in \Theta_{ad} \text{ and } u \in \mathcal{U}(\Omega), \end{cases} \tag{1}$$

where $\mathcal{U}(\Omega)$ denotes the set of solutions of a given partial differential equation on the domain Ω (state equation on Ω) and Θ_{ad} is a set of admissible domains.

In order to show the existence of a solution for this type of problems, a widely used method is to show that the space $\mathcal{F} = \{(\Omega, u)\ ,\ \Omega \in \Theta_{ad}\,,\ u \in \mathcal{U}(\Omega)\}$ is compact with a suitable topology on $\mathcal{F}$ which is induced from topologies on Θ_{ad} and $\mathcal{U}(\Omega)$, and that the cost functional J is lower semi continuous on $\mathcal{F}$ [23].

In this paper, we are interested in the continuity of cost functionals that can be associated to the determination of the location, size and shape of an unknown portion Γ, of the boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^2$, from given, Dirichlet and Neumann, boundary conditions on this unknown part of the boundary. More precisely, we consider the boundary cost functional J defined by $J(\Omega, u) = \displaystyle\int_\Gamma (u-g)^2\, d\sigma$, which is associated with problems where a condition of the type $u = g$ has to be imposed on Γ. The function g is given and u is the solution of a partial differential equation on Ω. In the case where the unknown boundary Γ is the graph of a function, continuity results for this type of functionals with a suitable topology on $\mathcal{F}$ have been obtained in [6, 10, 24]. However, in many physical problems this assumption on the unknown boundary is too restrictive. The situation where the unknown boundary can not be the graph of a function occurs in many engineering problems such as, for example, the dam problem in non-homogeneous porous media [10, 19, 22], the Stefan problem [20], optimal insulating and electro-chemistry [3, 17] and the semiconductor problem [27, 16]. Here, we use a general parameterization of the unknown boundary in order to preserve the physical general information on this boundary. The topology we use on Θ_{ad} is just that associated to the C^1 convergence of the parameterizations.

The main result of this paper is the continuity of the trace operator from $H^r(\Omega)$ $(1 \geq r > \frac{1}{2})$ to $L^2(\Gamma)$ with a constant independent of Γ. This is proved in section 3. At this stage, it should be noted that, in the case of a family of domains with parallel boundaries, a similar result has been obtained by Lions and Magenes in [30]. The continuity of the cost functional J on the space $\mathcal{F}$ is established in section 4. Actually, we prove the continuity of a more general functional.

2 Notations and definitions

Let D be a fixed C^1 open connected bounded subset of $\mathbb{R}^2$. An admissible domain Ω will be an open subset of D. The boundary $\partial\Omega$ of Ω is assumed to consist in two parts $\partial\Omega = \Gamma_0 \cup \Gamma$ such that $\Gamma_0 \cap \Gamma = \emptyset$, $meas(\Gamma_0) > 0$ and $meas(\Gamma) > 0$, where Γ is the free boundary part defined by

$$\Gamma = \Gamma(\varphi) = \{\varphi(t) = (\varphi_1(t), \varphi_2(t))\,;\quad t \in [0, 1]\},$$

$\varphi : [0,1] \to \mathbb{R}^2$ is a parameterisation and Γ is assumed to have its endpoints on the boundary of D (see figure 1). In fact, in this general setting, the curve $\Gamma(\varphi)$ can divide D into two open subsets. This is the case for example when D is simply connected. We have then to choose Ω. We can make for example the following choice : We follow the natural orientation given by the parameterisation and take the open set whose exterior unit normal boundary vector is on the left.

We shall also write $\Omega = \Omega(\varphi)$ to indicate the dependence on the parameterization φ.

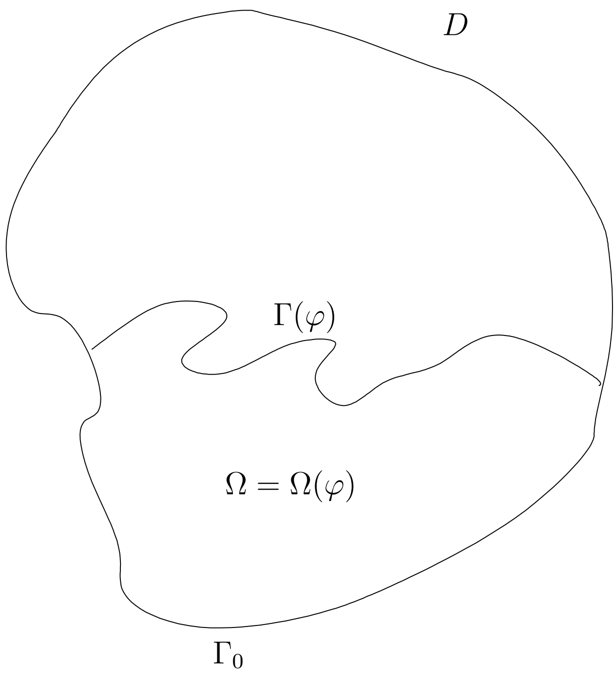

Figure 1: An example of the considered domain $\Omega = \Omega(\varphi)$.

We shall denote by $\nu(x)$ the exterior unit normal vector to ∂D at $x \in \partial D$.

Define V_{ad} to be the set of vector functions $\varphi \in C^1\left([0,1], \mathbb{R}^2\right)$ satisfying the following conditions :

$$|\varphi(t)| \leq C_0, \quad \forall t \in [0,1],$$

$$C_1|t-t'| \leq |\varphi(t) - \varphi(t')| \leq C_2|t-t'|, \quad \forall t, t' \in [0,1],$$

$$\varphi([0,1]) \subset \overline{D} \text{ and, for } t = 0 \text{ or } 1, \; \varphi(t) \in \partial D,$$

$$\text{for } t = 0 \text{ or } 1, \; |\nu(\varphi(t)).\varphi'(t)| \geq \epsilon_0, \tag{2}$$

$$\text{and, for all } t \in [0,1], \; d(\varphi(t), \partial D) \geq \epsilon_1 \, t(1-t), \tag{3}$$

where C_0, C_1, C_2, ϵ_0 and ϵ_1 are positive fixed constants, $d(\varphi(t), \partial D)$ is the distance of the point $\varphi(t)$ from the boundary ∂D, and we are using the standard notations $|(a,b)| = \sqrt{a^2+b^2}$, $(a,b).(a',b') = aa' + bb'$. Note that the assumptions (2) and (3) are made to insure that the curve $\Gamma(\varphi)$ touches the boundary ∂D only two times (with the endpoints) and in a transverse manner. The consequence of this is that the domain $\Omega(\varphi)$ is Lipschitz regular.

Clearly, V_{ad} is a closed and bounded subset of $C^1([0,1], \mathbb{R}^2)$.

Now, the set U_{ad} of admissible functions will be any compact subset of V_{ad}. In other words, U_{ad} is a subset of V_{ad} whose elements and their derivatives are equicontinuous as it follows from Ascoli-Arzelà theorem. An example of such a set U_{ad} is that of a closed subset of V_{ad} which is bounded in $C^{1,\delta}\left([0,1],\mathbb{R}^2\right)$ for some δ such that $0<\delta\leq 1$.

The space of admissible domains is then defined by :

$$\Theta_{ad} = \{\Omega = \Omega(\varphi) \subset D\,;\;\; \varphi \in U_{ad}\}\,.$$

Note that the elements of Θ_{ad} are uniformly Lipschitz open sets of $\mathbb{R}^2$ and so they satisfy the uniform cone property [34].

Assume that $u = u(\Omega) \in H^1(\Omega)$ is the solution of a well posed problem such as

$$\left\{\begin{array}{lll} L_{\Omega}\, u = f & \text{in} & \Omega \\ L_{\partial\Omega}\, u = h & \text{on} & \partial\Omega, \end{array}\right. \tag{4}$$

where $\Omega \in \Theta_{ad}$ and L_{Ω} and $L_{\partial\Omega}$ are partial differential operators on Ω and $\partial\Omega$ respectively.

The cost functional J we are interested in is then defined on the set

$$\mathcal{F} = \{(\Omega(\varphi), u(\Omega))\,;\;\; \Omega(\varphi) \in \Theta_{ad} \text{ and }\; u(\Omega)\; \text{ solves }\; (4) \text{ on } \Omega(\varphi)\}$$

by

$$J(\Omega(\varphi), u(\Omega)) = \int_{\Gamma(\varphi)} (u(\Omega) - g)^2\, d\sigma,$$

where g is a given function on D having essentially the same regularity as $u(\Omega)$. For simplicity and without any restriction on our study, we shall assume that $g = 0$.

Now, the customary problem of shape optimization or optimal shape design is

$$\text{to minimize }\; J(\Omega, u) \;\text{ on }\; \mathcal{F}. \tag{5}$$

As we have already said, this minimization problem is usually solved by endowing the set $\mathcal{F}$ with a topology for which $\mathcal{F}$ is compact and J is lower semi-continuous. Let us therefore define the topology we shall work with. First, we define the convergence of a sequence $(\varphi_n)_n \subset U_{ad}$ by

$$\varphi_n \;\longrightarrow\; \varphi \quad\Longleftrightarrow\quad \left\{\begin{array}{l} \varphi_n \longrightarrow \varphi \;\text{ uniformly on } [0,1] \\ \varphi_n' \longrightarrow \varphi' \;\text{ uniformly on } [0,1] \end{array}\right. \tag{6}$$

that is, iff $\varphi_n \to \varphi$ in the C^1 topology. Then, the convergence of a sequence $(\Omega_n)_n \subset \Theta_{ad}$, such that $\Omega_n = \Omega(\varphi_n)$, to $\Omega = \Omega(\varphi) \in \Theta_{ad}$ is simply defined by

$$\Omega_n \;\longrightarrow\; \Omega \quad\Longleftrightarrow\quad \varphi_n \;\longrightarrow\; \varphi. \tag{7}$$

Denote by $\widetilde{u}$ the uniform extension of u from Ω to an open smooth bounded domain B (a disc, for example), such that $\overline{D} \subset B$ (see [12]). We define the convergence of a sequence $(u_n)_n$ of solutions of (4) on $\Omega(\varphi_n)$ to u the solution of (4) on $\Omega(\varphi)$ by

$$u_n \longrightarrow u \quad\Longleftrightarrow\quad \widetilde{u}_n \rightharpoonup \widetilde{u} \;\text{ weakly in }\; H^1(B). \tag{8}$$

Finally, the topology we put on $\mathcal{F}$ is the one induced by the convergence defined by

$$(\Omega_n, u_n) \longrightarrow (\Omega, u) \iff \begin{cases} \Omega_n \longrightarrow \Omega \\ u_n \longrightarrow u. \end{cases} \tag{9}$$

Thus, the compactness of $\mathcal{F}$ with respect to this topology depends on the type of the state problem (4) and on the compactness of Θ_{ad} with respect to the convergence (7). At this stage, we have to say that it is not our objective here to prove the compactness of $\mathcal{F}$, the setting being too general. But since U_{ad} is compact, one can obtain the compactness of $\mathcal{F}$ from the continuity of the state problem (4), which is based on the following (reasonable) condition : There exists a constant $C > 0$ such that $\|u(\Omega)\|_{1,\Omega} \leq C$, $\forall \Omega \in \Theta_{ad}$. Anyhow, the chosen topology is rather natural, is used in many applied problems and, as we shall see, it allows to prove the continuity of the cost functional J and even the continuity of more general boundary functionals (see section 4).

3 Continuity of the trace operator

In this section, we give a proof of the trace theorem which allows us to estimate the norm of the trace operator by a constant independent of the free boundary Γ.

Theorem 1 *Let u be in $H^r(B)$, $1 \geq r > \frac{1}{2}$, and $\Omega(\varphi) \in \Theta_{ad}$. Then, there exists a constant K independent of φ such that*

$$\|u\|_{0,\Gamma(\varphi)} \leq K \, \|u\|_{r,B}$$

where $\|\cdot\|_{0,\Gamma(\varphi)}$ is the $L^2(\Gamma(\varphi))$-norm and $\|\cdot\|_{r,B}$ is the $H^r(B)$-norm.

The proof of this result is based on the following constructions and techniques.

Let $\varphi \in U_{ad}$. By using standard techniques, we can construct a continuous extension operator from $C^1([0,1], \mathbb{R}^2)$ to $\tilde{C}^1([-1,2], \mathbb{R}^2) = \{\varphi \in C^1(\mathbb{R}, \mathbb{R}^2);\ \mathrm{supp}(\varphi) \subset [-1,2]\}$, such that the extension $\widetilde{\varphi}$ of φ to $\mathbb{R}$ satisfies

$$\|\widetilde{\varphi}'\|_{L^\infty(\mathbb{R})} = \|\varphi'\|_{L^\infty([0,1])}, \tag{10}$$

where $\|\cdot\|_{L^\infty}$ is the L^∞-norm.

Now, let $\chi \in \mathcal{D}(\mathbb{R})$ such that $\chi \geq 0$, $\displaystyle\int_{\mathbb{R}} \chi \, dx = 1$ and $\mathrm{supp}(\chi) \subset [-1,1]$. We define the function $\psi_k = (\psi_{1,k}, \psi_{2,k})$, $k \in \mathbb{N}^*$, on $\mathbb{R}$ as a regularized function of $\widetilde{\varphi}$, that is,

$$\psi_k(t) = \int_{\mathbb{R}} \widetilde{\varphi}(t-\tau) \, \chi(k\tau) \, k \, d\tau.$$

Note that

$$\psi_k'(t) = \int_{\mathbb{R}} \widetilde{\varphi}'(t-\tau) \, \chi(k\tau) \, k \, d\tau.$$

Since $\widetilde{\varphi}'$ is continuous, it is well known that ψ_k' converges uniformly to $\widetilde{\varphi}'$ on any compact subset of $\mathbb{R}$ as $k \longrightarrow \infty$. In particular,

$$\lim_{k \longrightarrow \infty} \|\psi_k' - \varphi'\|_{L^\infty([0,1])} = 0.$$

In fact, we can show that for any $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}^*$ independent of φ such that, for $k \geq k_\varepsilon$,

$$\|\psi_k' - \varphi'\|_{L^\infty([0,1])} < \varepsilon.$$

Indeed, we have

$$\begin{aligned}|\psi'_k(t)-\widetilde{\varphi}'(t)| &= |\int_{\mathbb{R}}(\widetilde{\varphi}'(t-\tau)-\widetilde{\varphi}'(t))\ \chi(k\tau)\,k\,d\tau| \\ &\leq \int_{\mathbb{R}}|\widetilde{\varphi}'(t-\frac{\tau}{k})-\widetilde{\varphi}'(t)|\,\chi(\tau)\,d\tau.\end{aligned}$$

Since the extension operator is continuous from $C^1\left([0,1],\mathbb{R}^2\right)$ to $\widetilde{C}^1\left([-1,2],\mathbb{R}^2\right)$, the functions $\widetilde{\varphi}'$, are equicontinuous and uniformly continuous when φ describes U_{ad}. So, for $\varepsilon>0$, there exists $\gamma_\varepsilon>0$ independent of φ such that

$$\forall t,t'\in\mathbb{R},\ \ |t-t'|\leq\gamma_\varepsilon \quad \text{implies that} \quad |\widetilde{\varphi}'(t)-\widetilde{\varphi}'(t')|\leq\varepsilon,\ \ \forall\varphi\in U_{ad}. \tag{11}$$

Hence, for k such that $\dfrac{|\tau|}{k}\leq\dfrac{1}{k}\leq\gamma_\varepsilon$ and all $t\in\mathbb{R}$,

$$|\psi'_k(t)-\widetilde{\varphi}'(t)|\leq\int_{\mathbb{R}}\varepsilon\,\chi(\tau)\,d\tau=\varepsilon.$$

So, we can take $k_\varepsilon=[\dfrac{1}{\gamma_\varepsilon}]+1$, where $[\,\cdot\,]$ denotes the integral part.

Now, to the given $\varphi\in U_{ad}$, we associate the function $\Phi\in C^1\left([0,1]\times\mathbb{R},\mathbb{R}^2\right)$ defined by

$$\Phi(t,s)=\varphi(t)+s\,\psi'_{k_\varepsilon}(t)^\perp\,,\ \ s\in\mathbb{R}, \tag{12}$$

where we are using the notation $(a,b)^\perp=(-b,a)$. In what follows, we shall omit the index k_ε and write ψ instead of ψ_{k_ε}.

The following lemma is basic for the proof of Theorem 1.

Lemma 1 *There exists a small enough $s_0>0$ such that s_0 is independent of $\varphi\in U_{ad}$ and the following three assertions hold.*

(i) *The Jacobian, $J\Phi$, of Φ is such that*

$$|J\Phi|\geq\frac{1}{2}C_1^2\ \ on\ \ [0,1]\times[-s_0,s_0]\,. \tag{13}$$

(ii) *There exists $C_3>0$ independent of φ such that*

$$|\Phi(t,s)-\Phi(t',s')|\leq C_3\,|(t-t',s-s')|,\qquad\forall(t,s),\ (t',s')\in[0,1]\times[-s_0,s_0]. \tag{14}$$

(iii) *Φ is injective in $[0,1]\times[-s_0,s_0]$, and, more precisely,*

$$|\Phi(t,s)-\Phi(t',s')|\geq\frac{\sqrt{2}}{4}C_1\,|(t-t',s-s')|,\qquad\forall(t,s),\ (t',s')\in[0,1]\times[-s_0,s_0]. \tag{15}$$

where C_1 is the same constant as that in the definition of U_{ad}.

Proof :

(i) We have that

$$\Phi'=\begin{pmatrix}\varphi'_1-s\,\psi''_2 & -\psi'_2\\ \varphi'_2+s\,\psi''_1 & \psi'_1\end{pmatrix}.$$

So,

$$\begin{aligned} J\Phi = det(\Phi') &= \varphi_1'\,\psi_1' + \varphi_2'\,\psi_2' + s\,(\psi_1''\,\psi_2' - \psi_1'\,\psi_2'') \\ &= |\varphi'|^2 + \varphi'\cdot(\psi-\varphi)' - s\,\psi''\cdot\psi'^{\perp}. \end{aligned}$$

It follows from the definition of $\psi = \psi_{k_\varepsilon}$ that

$$\|\psi'^{\perp}\|_{L^\infty([0,1])} \le \|\psi'\|_{L^\infty(\mathbb{R})} \le \|\widetilde{\varphi}'\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \chi(k_\varepsilon\tau)\,k_\varepsilon\,d\tau \|\widetilde{\varphi}'\|_{L^\infty([0,1])} \le C_2 \tag{16}$$

and

$$\|\psi''\|_{L^\infty([0,1])} \le \|\psi''\|_{L^\infty(\mathbb{R})} \le \|\widetilde{\varphi}'\|_{L^\infty(\mathbb{R})}\,k_\varepsilon^2 \int_{\mathbb{R}} |\chi'(k_\varepsilon\tau)|\,d\tau \le C_2\,k_\varepsilon \int_{\mathbb{R}} |\chi'(\tau)|\,d\tau \equiv C_2' k_\varepsilon. \tag{17}$$

Using equations (16), (17) and the inequality $C_1 \le \|\varphi'\|_{L^\infty([0,1])} \le C_2$, which follows from the definition of U_{ad}, we obtain

$$|J\Phi| \ge C_1^2 - C_2\,\varepsilon - |s|\,C_2\,C_2' k_\varepsilon \ge C_1^2 - C_2\,(\varepsilon + s_0\,C_2' k_\varepsilon), \quad \forall s,\ |s| \le s_0.$$

Finally, choosing ε and s_0 so small that

$$C_2\,(\varepsilon + s_0\,C_2' k_\varepsilon) \le \frac{1}{2} C_1^2,$$

we obtain

$$|J\Phi| \ge \frac{1}{2} C_1^2 \quad \text{on} \quad [0,1]\times[-s_0,s_0]\,.$$

(ii) We have, for all $(t,s) \in [0,1]\times[-s_0,s_0]$,

$$\begin{aligned} |\varphi'(t) + s\,\psi''(t)^{\perp}| &\le \|\varphi'\|_{L^\infty([0,1])} + s\,\|\psi''\|_{L^\infty([0,1])} \\ &\le C_2 + s_0\,C_2' k_\varepsilon, \end{aligned}$$

and

$$|\psi'(t)^{\perp}| \le \|\psi'\|_{L^\infty([0,1])} \le C_2.$$

Hence, if $C_3 C_2(1 + s_0 k_\varepsilon \|\chi'\|_{L^1})$, we obtain, by Taylor formula,

$$|\Phi(t,s) - \Phi(t',s')| \le C_3\,|(t-t', s-s')|,$$

for all (t,s), $(t',s') \in [0,1]\times[-s_0,s_0]$.

(iii) Let us now show that Φ is injective in $[0,1]\times[-s_0,s_0]$, for s_0 small enough and independent of φ. To this end, we shall show that there exists a constant M such that

$$|\Phi(t,s) - \Phi(t',s')| \ge M\,|(t-t', s-s')|, \quad \forall (t,s),\ (t',s') \in [0,1]\times[-s_0,s_0].$$

Let (t,s), $(t',s') \in [0,1]\times[-s_0,s_0]$. We have

$$\begin{aligned} \Phi(t,s) - \Phi(t',s') &= \varphi(t) - \varphi(t') + s\,\psi'(t)^{\perp} - s'\,\psi'(t')^{\perp} \\ &= \varphi(t) - \varphi(t') + (s-s')\,\psi'(t)^{\perp} + s'(\psi'(t)^{\perp} - \psi'(t')^{\perp}). \end{aligned} \tag{18}$$

Now, we discuss two cases :

First case, if $|s-s'| \le \eta\,|t-t'|$, η small enough, we have

$$\begin{array}{rcl} |\Phi(t,s)-\Phi(t',s')| & \ge & C_1\,|t-t'| - \eta\,|t-t'|\,\|\psi'\|_{L^\infty([0,1])} - s_0\,\|\psi''\|_{L^\infty([0,1])}\,|t-t'| \\ & \ge & (C_1 - \eta\,\|\psi'\|_{L^\infty([0,1])} - s_0\,\|\psi''\|_{L^\infty([0,1])})\,|t-t'| \\ & \ge & (C_1 - \eta\,C_2 - s_0\,C_2' k_\varepsilon)\,|t-t'| \\ & \ge & (C_1 - \eta\,C_2 - s_0\,C_2' k_\varepsilon)\,\left(\frac{1}{2}\,|t-t'|^2 + \frac{1}{2}\,\frac{1}{\eta^2}\,|s-s'|^2\right)^{\frac{1}{2}}. \end{array}$$

Assuming that $\eta \le 1$, we obtain

$$\|\Phi(t,s)-\Phi(t',s')\| \ge \frac{1}{\sqrt{2}}\,(C_1 - \eta\,C_2 - s_0\,C_2' k_\varepsilon)\,|(t-t',s-s')|.$$

Second case, if $|s-s'| \ge \eta\,|t-t'|$, we can rewrite the terms $\varphi(t)-\varphi(t')$ and $(s-s')\,\psi'(t)^\perp$ in (18) as

$$\varphi(t')-\varphi(t) = (t'-t)\,\varphi'(t) + (t-t')\,\int_0^1 (\varphi'(t+\tau(t-t')) - \varphi'(t))\;d\tau$$

and

$$(s-s')\,\psi'^{\perp}(t) = (s-s')\,(\varphi'(t))^\perp + (s-s')\;(\psi'(t)-\varphi'(t))^\perp\,.$$

We know that

$$\|\psi'-\varphi'\|_{L^\infty([0,1])} \le \varepsilon$$

and, from (11), that there exists $\gamma_\varepsilon > 0$ independent of φ such that

$$|t-t'| \le \gamma_\varepsilon \quad \text{implies that} \quad |\varphi'(t+\tau(t-t')-\varphi'(t)| \le \varepsilon,\;\; \forall \tau \in [0,1]. \tag{19}$$

Hence,

$$|s-s'| \le \eta\,\gamma_\varepsilon \quad \text{implies that} \quad |\int_0^1 (\varphi'(t+\tau(t-t')-\varphi'(t))\;d\tau| \le \varepsilon. \tag{20}$$

Thus, for $s_0 \le \frac{1}{2}\,\eta\,\gamma_\varepsilon$, we have

$$\begin{array}{rl} & |\Phi(t,s)-\Phi(t',s')| \\ \ge & |-(t-t')\,\varphi'(t) + (s-s')\,\varphi'(t)^\perp| - \varepsilon\,|t-t'| - \varepsilon\,|s-s'| - s_0\,\|\psi''\|_{L^\infty([0,1])}\,|t-t'| \\ \ge & \left(|t-t'|^2\,|\varphi'(t)|^2 + |s-s'|^2\,|\varphi'(t)|^2\right)^{\frac{1}{2}} - (2\varepsilon + s_0\,C_2' k_\varepsilon)\,|(t-t',s-s')| \\ \ge & (C_1 - 2\,\varepsilon - s_0\,C_2' k_\varepsilon)|(t-t',s-s')|. \end{array}$$

The constants ε, η and s_0 can be chosen such that, for example,

$$\eta\,C_2 + s_0\,C_2' k_\varepsilon \le \frac{C_1}{2},\quad 2\,\varepsilon + s_0\,C_2' k_\varepsilon \le \frac{C_1}{2} \text{ and } s_0 \le \frac{1}{2}\,\eta\,\gamma_\varepsilon.$$

Hence, it suffices to take

$$\eta = \frac{C_1}{4\,C_2},\quad \varepsilon = \frac{C_1}{8} \text{ and } s_0 = \min\left\{\frac{C_1\gamma_\varepsilon}{8C_2}, \frac{C_2}{8C_2 k_\varepsilon \|\chi\|_{L^1}}\right\}.$$

This shows that there exists s_0 independent of φ, such that Φ is injective in $[0,1]\times[-s_0,s_0]$ and

$$|\Phi(t,s)-\Phi(t',s')| \ge \frac{\sqrt{2}}{4} C_1\,|(t-t',s-s')|, \qquad \forall (t,s),\,(t',s') \in [0,1]\times[-s_0,s_0].$$

$\blacksquare$

<u>**Proof of theorem 1 :**</u>

Let us denote by $\mathcal{I} =]0,1[$ and $\mathcal{J} =]-s_0, s_0[$ and let us consider $u \in H^r(B)$ ($\frac{1}{2} < r \leq 1$). We have that

$$u(\varphi(t)) = u_{|\Gamma} \circ \varphi(t), \quad \forall t \in [0,1],$$

and, on the other hand,

$$u(\varphi(t)) = u(\Phi(t,s))_{|s=0} \equiv v(t,s)_{|s=0}, \quad \forall t \in [0,1],$$

where Φ is the function defined in (12) and $v = u \circ \Phi$. From the above lemma, we have that Φ is a C^1 diffeomorphism from $\mathcal{I} \times \mathcal{J}$ onto an open set of $\mathbb{R}^2$ which is some tubular neighborhood of Γ, and thus $v \in H^r(\mathcal{I} \times \mathcal{J})$. Now,

$$\|u\|_{0,\Gamma(\varphi)} \leq C_2 \, \|u \circ \varphi\|_{0,\mathcal{I}} = C_2 \, \|v(t,s)_{|s=0}\|_{0,\mathcal{I}},$$

and, according to the standard result on the continuity of the trace operator from $H^r(\mathcal{I} \times \mathcal{J})$ to $L^2(\mathcal{I} \times \{0\})$ (for $r > \frac{1}{2}$), there exists a constant β, independent of v, such that $\|v|_{s=0}\|_{0,\mathcal{I}} \leq \beta \|v\|_{r,\mathcal{I}\times\mathcal{J}}$ for all $v \in H^r(\mathcal{I} \times \mathcal{J})$. Hence,

$$\|u\|_{0,\Gamma(\varphi)} \leq C_2 \, \beta \, \|v\|_{r,\mathcal{I}\times\mathcal{J}}.$$

Next, there exists a constant C_4 independent of φ, such that

$$\|v\|_{r,\mathcal{I}\times\mathcal{J}} \leq C_4 \, \|u\|_{r,B}.$$

Indeed, we have

$$\int\int_{\mathcal{I}\times\mathcal{J}} |u(\Phi(t,s))|^2 \, dt \, ds = \int\int_{\widetilde{\Omega}} |u(x,y)|^2 \, |det((\Phi^{-1})'(x,y)| \, dx \, dy$$

where $\widetilde{\Omega} = \Phi(\mathcal{I} \times \mathcal{J})$ and $(x,y) = \Phi(t,s)$, and it follows from Lemma 1 that

$$\begin{aligned}
\int\int_{\mathcal{I}\times\mathcal{J}} |u(\Phi(t,s))|^2 \, dt \, ds &= \int\int_{\widetilde{\Omega}} |u(x,y)|^2 \, \frac{dx \, dy}{|det\Phi'(\Phi^{-1}(x,y))|} \\
&\leq \frac{2}{C_1^2} \int\int_{\widetilde{\Omega}} |u(x,y)|^2 \, dx \, dy \\
&\leq \frac{2}{C_1^2} \, \|u\|^2_{0,\widetilde{\Omega}}.
\end{aligned}$$

Thus,

$$\|u \circ \Phi\|^2_{0,\mathcal{I}\times\mathcal{J}} \leq \frac{2}{C_1^2} \, \|u\|^2_{0,\widetilde{\Omega}}. \tag{21}$$

On the other hand, setting $\mathcal{U} = \mathcal{I} \times \mathcal{J}$, we have from Lemma 1

$$\begin{aligned}
&\int\int_{\mathcal{U}\times\mathcal{U}} \frac{|u \circ \Phi(t,s) - u \circ \Phi(t',s')|^2}{|(t,s) - (t',s')|^{2r+2}} \, dt \, dt' \, ds \, ds' \\
= &\int\int_{\widetilde{\Omega}\times\widetilde{\Omega}} \frac{|u(x,y) - u(x',y')|^2}{|\Phi^{-1}(x,y) - \Phi^{-1}(x',y')|^{2r+2}} \times |det((\Phi^{-1})'(x,y)| \, |det((\Phi^{-1})'(x',y')| \, dx \, dx' \, dy \, dy' \\
\leq &\frac{4}{C_1^4} \int\int_{\widetilde{\Omega}\times\widetilde{\Omega}} \frac{|u(x,y) - u(x',y')|^2}{|\Phi^{-1}(x,y) - \Phi^{-1}(x',y')|^{2r+2}} \, dx \, dx' \, dy \, dy',
\end{aligned}$$

Since Φ is Lipschitz of constant C_3, we have that

$$|\Phi^{-1}(x,y) - \Phi^{-1}(x',y')| \geq \frac{1}{C_3} |(x-x', y-y')| \quad \forall (x,y),\, (x',y') \in \widetilde{\Omega} \times \widetilde{\Omega}. \tag{22}$$

Therefore,

$$\begin{aligned} & \int\int_{\mathcal{U}\times\mathcal{U}} \frac{|u\circ\Phi(t,s) - u\circ\Phi(t',s')|^2}{|(t,s)-(t',s')|^{2r+2}}\, dt\, dt'\, ds\, ds' \\ \leq\ & \frac{4}{C_1^4} C_3^{2+2r} \int\int_{\widetilde{\Omega}\times\widetilde{\Omega}} \frac{|u(x,y)-u(x',y')|^2}{|(x,y)-(x',y')|^{2r+2}}\, dx\, dx'\, dy\, dy'. \\ \leq\ & \frac{4\, C_3^{2+2r}}{C_1^4} \|u\|_{r,B}^2 \end{aligned} \tag{23}$$

(21) and (23)imply the theorem when $r < 1$.

When $r = 1$, we estimate the partial derivatives of in the same manner (one can also estimate $\|u\|_{r,B}$ by $\|u\|_{1,B}$). This achieves the proof of the theorem. ■

Now, as a consequence of Theorem 1, we state and prove the following convergence result which essentially says that $u|_{\Gamma_n} \longrightarrow u|_\Gamma$. It will also be needed in next section.

Corollary 1 *Let $(\varphi_n)_n \subset U_{ad}$ be a sequence such that $\varphi_n \to \varphi$ in the sense of (6), that is for the C^1 convergence, and let $u \in H^1(B)$. Then*

$$\lim_{n\longrightarrow\infty} u\circ\varphi_n u\circ\varphi \quad \text{in } L^2([0,1]).$$

<u>Proof :</u>

First, it follows from the density of $\mathcal{D}(\overline{B})$ in $H^1(B)$ (see [**?**]) that, for a given $\varepsilon > 0$, there exists $\upsilon_\varepsilon \in \mathcal{D}(\overline{B})$ such that

$$\|\upsilon_\varepsilon - u\|_{1,B} \leq \frac{\varepsilon}{3\sqrt{C_1}\, K},$$

where K is the constant of Theorem 1. Next, we can write

$$\begin{aligned} \|u\circ\varphi_n - u\circ\varphi\|_{0,[0,1]} &= \|u\circ\varphi_n - \upsilon_\varepsilon\circ\varphi_n + \upsilon_\varepsilon\circ\varphi_n - \upsilon_\varepsilon\circ\varphi + \upsilon_\varepsilon\circ\varphi - u\circ\varphi\|_{0,[0,1]} \\ &\leq \|u\circ\varphi_n - \upsilon_\varepsilon\circ\varphi_n\|_{0,[0,1]} + \|\upsilon_\varepsilon\circ\varphi_n - \upsilon_\varepsilon\circ\varphi\|_{0,[0,1]} \\ &\qquad + \|\upsilon_\varepsilon\circ\varphi - u\circ\varphi\|_{0,[0,1]}. \end{aligned} \tag{24}$$

Now, by Theorem 1,

$$\|\upsilon_\varepsilon\circ\varphi_n - u\circ\varphi_n\|_{0,[0,1]} = \left(\int_0^1 |\upsilon_\varepsilon\circ\varphi_n(t) - u\circ\varphi_n(t)|^2\, dt\right)^{\frac{1}{2}} \leq \sqrt{C_1}\, K\, \|\upsilon_\varepsilon - u\|_{1,B},$$

and according to the Lebesgue convergence theorem, we have

$$\lim_{n\longrightarrow\infty} \|\upsilon_\varepsilon\circ\varphi_n - \upsilon_\varepsilon\circ\varphi\|_{0,[0,1]} = \lim_{n\longrightarrow\infty} \left(\int_0^1 |\upsilon_\varepsilon\circ\varphi_n(t) - \upsilon_\varepsilon\circ\varphi(t)|^2\, dt\right)^{\frac{1}{2}} = 0$$

Thus,

$$\|\upsilon_\varepsilon\circ\varphi_n - u\circ\varphi_n\|_{0,[0,1]} = \left(\int_0^1 |\upsilon_\varepsilon\circ\varphi_n(t) - u\circ\varphi_n(t)|^2\, dt\right)^{\frac{1}{2}} \leq \sqrt{C_1}\, K\, \|\upsilon_\varepsilon - u\|_{1,B} \leq \frac{\varepsilon}{3}, \tag{25}$$

$$\|v_\varepsilon \circ \varphi - u \circ \varphi\|_{0,[0,1]} = \left(\int_0^1 |v_\varepsilon \circ \varphi(t) - u \circ \varphi(t)|^2 \, dt\right)^{\frac{1}{2}} \leq \sqrt{C_1}\, K \, \|v_\varepsilon - u\|_{1,B} \leq \frac{\varepsilon}{3}, \tag{26}$$

and there exists $N \in \mathbb{N}^*$ such that $n \geq N$ implies

$$\|v_\varepsilon \circ \varphi_n - v_\varepsilon \circ \varphi\|_{0,[0,1]} = \left(\int_0^1 |v_\varepsilon \circ \varphi_n(t) - v_\varepsilon \circ \varphi(t)|^2 \, dt\right)^{\frac{1}{2}} \leq \frac{\varepsilon}{3}. \tag{27}$$

Hence, for $n \geq N$,

$$\|u \circ \varphi_n - u \circ \varphi\|_{0,[0,1]} = \left(\int_0^1 |u \circ \varphi_n(t) - u \circ \varphi(t)|^2 \, dt\right)^{\frac{1}{2}} \leq \varepsilon, \tag{28}$$

which ends the proof of the corollary. ■

4 Applications

We now apply Theorem 1 to study the continuity of the boundary cost functional

$$J(\Omega, u) = \int_\Gamma |u|^2 d\sigma$$

which appears in many problems of optimal shape design, often as a shape optimization formulation of the Dirichlet type condition of a free boundary problem. Of course, Theorem 2 below can also be considered as a continuity result for the trace operator with respect to the couple (u, Γ).

Theorem 2 *The functional J is continuous on $\mathcal{F}$ with the topology induced by the convergence defined in (9).*

Proof :
Let $\{(\Omega_n, u_n)\}_n$ be a sequence of $\mathcal{F}$, such that $\Omega_n = \Omega(\varphi_n)$, $\Omega = \Omega(\varphi)$ and

$$(\Omega_n, u_n) \longrightarrow (\Omega, u) \quad \text{as } n \longrightarrow \infty\,.$$

In what follows, the functions under consideration are of course the extensions $\widetilde{u}$, $\widetilde{u}_n \in H^1(B)$, but for simplicity we shall drop the "$\widetilde{\ }$". To show that $J(\Omega_n, u_n) \longrightarrow J(\Omega, u)$, let us prove that $\sqrt{J(\Omega_n, u_n)} \longrightarrow \sqrt{J(\Omega, u)}$. We have:

$$\begin{aligned}
& \left|\sqrt{J(\Omega_n, u_n)} - \sqrt{J(\Omega, u)}\right| \\
= \; & \left| \|u_n \circ \varphi_n \cdot |\varphi_n'|^{\frac{1}{2}}\|_{0,[0,1]} - \|u \circ \varphi \cdot |\varphi'|^{\frac{1}{2}}\|_{0,[0,1]} \right| \\
\leq \; & \left\| u_n \circ \varphi_n \cdot |\varphi_n'|^{\frac{1}{2}} - u \circ \varphi \cdot |\varphi'|^{\frac{1}{2}} \right\|_{0,[0,1]} \\
\leq \; & \left\| (u_n \circ \varphi_n - u \circ \varphi_n)\, |\varphi_n'|^{\frac{1}{2}} \right\|_{0,[0,1]} + \left\| (u \circ \varphi_n - u \circ \varphi)\, |\varphi_n'|^{\frac{1}{2}} \right\|_{0,[0,1]} \\
& + \left\| u \circ \varphi \left(|\varphi_n'|^{\frac{1}{2}} - |\varphi'|^{\frac{1}{2}} \right) \right\|_{0,[0,1]} \\
\leq \; & \|u_n - u\|_{0,\Gamma_n} + \sqrt{C_2}\, \|(u \circ \varphi_n - u \circ \varphi)\|_{0,[0,1]} + \frac{1}{\sqrt{C_1}} \|u\|_{0,\Gamma} \sup_{[0,1]} \left| |\varphi_n'|^{\frac{1}{2}} - |\varphi'|^{\frac{1}{2}} \right| \\
\leq \; & \|u_n - u\|_{r,B} + \sqrt{C_2}\, \|(u \circ \varphi_n - u \circ \varphi)\|_{0,[0,1]} + \frac{1}{2C_1} \|u\|_{0,\Gamma} \sup_{[0,1]} |\varphi_n' - \varphi'|\,, \qquad (29)
\end{aligned}$$

$\frac{1}{2} < r < 1$, by Theorem 1. The theorem follows from Corollary 1 and the compactness of the injection of $H^1(B)$ into $H^r(B)$. ■

Last, we establish the following more general continuity result. It concerns a general boundary cost functional of the form

$$J(\Omega, u) = \int_\Gamma f(x, u(x))d\sigma \tag{30}$$

where f is a real continuous function defined in $\mathbb{R}^2 \times \mathbb{R}$ (or $\mathbb{R}^2 \times \mathbb{C}$ if u is allowed to take complex values) such that

$$|f(x, u)| \leq C\left(1 + |u|^2\right) \tag{31}$$

for all x and u, C being a positive constant. Note that it follows from (31) that J is well defined on $\mathcal{F}$.

Theorem 3 *Under the above assumptions, J is continuous on $\mathcal{F}$ with the topology induced by the convergence defined in (9).*

Proof :

Taking the same sequence (Ω_n, u_n) as in the beginning of the proof of Theorem 2, we can write

$$J(\Omega_n, u_n) I_1(n) + I_2(n) + J(\Omega, u)$$

where

$$I_1(n) = \int_0^1 \left[f(\varphi_n(t), u_n(\varphi_n(t))) - f(\varphi(t), u(\varphi(t)))\right] |\varphi_n'(t)|\, dt,$$

$$I_2(n) = \int_0^1 f(\varphi(t), u(\varphi(t)))\ \left(|\varphi_n'(t)| - |\varphi'(t)|\right)\, dt.$$

$I_2(n)$ is easy to estimate. Indeed, obviously,

$$|I_2(n)| \leq C \int_0^1 (1 + |u(\varphi(t))|^2)dt\ \sup_{[0,1]} |\varphi_n' - \varphi'|\,,$$

so,

$$\lim_{n \longrightarrow \infty} I_2(n) = 0.$$

To treat $I_1(n)$, we need the following lemma which improves Corollary 1.

Lemma 2 *Let $(\varphi_n) \subset U_{ad}$ be a sequence such that $\varphi_n \longrightarrow \varphi \in U_{ad}$ in the sense of (6), and let $u, u_n \in H^1(B)$ such that u_n converges weakly to u in $H^1(B)$. Then, $u_n \circ \varphi_n \longrightarrow u \circ \varphi$ in $L^2([0, 1])$.*

Proof :

We have

$$\begin{aligned}
\|u_n \circ \varphi_n - u \circ \varphi\|_{0,[0,1]} &\leq \|u_n \circ \varphi_n - u \circ \varphi_n\|_{0,[0,1]} + \|u \circ \varphi_n - u \circ \varphi\|_{0,[0,1]} \\
&\leq \frac{1}{\sqrt{C_1}} \|u_n - u\|_{0,\Gamma_n} + \|(u \circ \varphi_n - u \circ \varphi)\|_{0,[0,1]} \\
&\leq \frac{K}{\sqrt{C_1}} \|u_n - u\|_{r,B} + \|(u \circ \varphi_n - u \circ \varphi)\|_{0,[0,1]}\,,
\end{aligned}$$

$\Gamma_n = \Gamma(\varphi_n)$, $\frac{1}{2} < r < 1$, by Theorem 1. Thus, the lemma follows from Corollary 1 and the compactness of the canonical injection of $H^1(B)$ into $H^r(B)$. ■

End of proof of Theorem 3: Since $u_n \circ \varphi_n \longrightarrow u \circ \varphi$ in $L^2([0,1])$, there exists a subsequence $u_{n_k} \circ \varphi_{n_k}$ and $v \in L^2([0,1])$ such that (see[7])

$$u_{n_k} \circ \varphi_{n_k} \longrightarrow u \circ \varphi, \ \ a.e. \text{ and } |u_{n_k} \circ \varphi_{n_k}| \leq v, \ \ a.e. \tag{32}$$

Consider now $I_1(n_k)$. We have :

- $[f(\varphi_{n_k}(t), u_{n_k}(\varphi_{n_k}(t))) - f(\varphi(t), u(\varphi(t)))] \, |\varphi'_{n_k}(t)| \longrightarrow 0, \ \ a.e.$
- $|f(\varphi_{n_k}(t), u_{n_k}(\varphi_{n_k}(t))) - f(\varphi(t), u(\varphi(t)))| \, |\varphi'_{n_k}(t)| \leq C_2 C(2 + v(t)^2 + |u(\varphi(t))|^2), \ \ a.e.$

So, it follows from the Lebesgue convergence theorem that $I_1(n_k) \longrightarrow 0$. However, this does not allow us to conclude. Therefore, consider $\underline{\lim} J(\Omega_n, u_n)$. Since $(J(\Omega_n, u_n))$ is a bounded real sequence, this limit exists and is equal to $\lim J(\Omega_{n_k}, u_{n_k})$ for some subsequence. Now,

$$\underline{\lim} J(\Omega_n, u_n) = \lim J(\Omega_{n_k}, u_{n_k}) = \lim J(\Omega_{n_{k_l}}, u_{n_{k_l}})$$

where (n_{k_l}) is such that the subsequence $u_{n_{k_l}} \circ \varphi_{n_{k_l}}$ satisfies (32). Hence, by the argument given above,

$$\underline{\lim} J(\Omega_n, u_n) = \lim J(\Omega_{n_{k_l}}, u_{n_{k_l}}) = J(\Omega, u).$$

Of course, the same argument works for $\overline{\lim} J(\Omega_n, u_n)$, and this proves the theorem. ■

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